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Theorems on Symmetries and Flux Conservation in Radiative Transfer

Using the Matrix Operator Theory

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by

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Abstract

The matrix operator approach to radiative transfer is shown to be a very powerful technique in establishing symmetry relations for multiple scattering in inhomogeneous atmospheres. Symmetries are derived for the reflection and transmission operators using only the symmetry of the phase function. These results will mean large savings in computer time and storage for performing calculations for realistic planetary atmospheres using this method. The results have also been extended to establish a condition on the reflection matrix of a boundary in order to preserve reciprocity. Finally energy conservation is rigorously proven for conservative scattering in inhomogeneous atmospheres.

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INTRODUCTION

The matrix operator theory of radiative transfer offers a very powerful approach to obtaining a complete analysis of the diffusion of radiation in a scattering, absorbing, and thermally radiating medium. The history of the method is quite fascinating and a brief account of it may be found in the article by Plass, Kattawar, and Catchings[1] hereafter referred to as PKC. If one is to properly utilize the method then one must make use of every symmetry available so that computation and core storage will be minimized.

Early proofs of symmetry relations have been done by Chandrasekhar,[2] Sekera[3], Busbridge[4], and A'Hearn[5]. The most recent analysis employing symmetries in the phase matrix has been performed by Hovenier[6]. He established certain symmetry relations for the reflection and transmission matrices for both homogeneous and inhomogeneous atmospheres.

It is the purpose of this paper to establish symmetry relations for the reflection and transmission operators used in the matrix operator theory. The proofs are completely different and simpler than those of Hovenier. Polarization will be neglected and only the scalar equations will be used. Relations will be derived for both homogeneous and inhomogeneous atmospheres. The effect of a reflecting surface on the symmetry properties will be derived for the first time. Finally energy conservation will be rigorously established in the matrix operator equations for a conservative scattering atmosphere.

II

Consider two regions of a plane parallel atmosphere bounded by planes located at (x,y) and (y,z) (see Figure 1a) where $x \leq y \leq z$. The reflection operator r and the transmission operator t for the combined layer are (see PKC):

$$r(x,z) = r(x,y) + t(y,x)\ [E - r(y,z)r(y,x)]^{-1}\ r(y,z)\ t(x,y), \tag{1a}$$

$$r(z,x) = r(z,y) + t(y,z)[E - r(y,x)r(y,z)]^{-1}\ r(y,x)t(z,y), \tag{1b}$$

$$t(x,z) = t(y,z)\ [E - r(y,x)r(y,z)]^{-1}\ t(x,y) \tag{2a}$$

$$t(z,x) = t(y,x)\ [E - r(y,z)r(y,x)]^{-1}\ t(z,y) \tag{2b}$$

where E denotes the identity operator. Continuous indices are now being used since the differential equations governing these operators will soon be derived. Denoting the radiance exiting from the top by $I^-(x)$ and that exiting out the bottom by $I^+(z)$ we have the following relations relating input to output:

$$I^-(x) = r(x,z)\ I^+(x) + t(z,x)\ I^-(z), \tag{3a}$$

$$I^+(z) = t(x,z)\ I^+(x) + r(z,x)\ I^-(z). \tag{3b}$$

where we have assumed no internal sources. It should be noted at this point that the operator $r(x,z)$ and $r(z,x)$ are in general not equal for inhomogeneous atmospheres and the same is true for $t(x,z)$ and $t(z,x)$.

In order to derive the differential equations governing r and t let us first consider the addition of a layer of thickness Δ to the bottom of the layer (x,y) (see Figure 1b). Equation (1b) now becomes

$$r(y+\Delta,x) = r(y+\Delta,y) + t(y,y+\Delta)\ [E-r(y,x)\ r(y,y+\Delta)]^{-1}\ r(y,x)\ t(y+\Delta,y) \tag{4}$$

where

$$r(y+\Delta,y) = \Gamma^{-+}(\xi)\Delta + O(\Delta), \tag{5a}$$

$$r(y,y+\Delta) = \Gamma^{+-}(\xi)\Delta + O(\Delta), \tag{5b}$$

$$t(y+\Delta,y) = E-\Gamma^{++}(\xi)\Delta + O(\Delta). \tag{5c}$$

$$t(y,y+\Delta) = E-\Gamma^{--}(\xi)\Delta + O(\Delta) \tag{5d}$$

where $y \leq \xi \leq y + \Delta$ and $O(\Delta)$ denotes a term such that $\lim\limits_{\Delta\to 0} \frac{O(\Delta)}{\Delta} \to 0$.

The definitions of Γ^{++}, Γ^{--}, Γ^{-+}, and Γ^{+-} will be introduced later in the sequel. Expanding both sides of eqn. (4), dividing by Δ, taking the limit as $\Delta \to 0$, and using eqns. (5a) and (5b) we get

$$\frac{\partial r(y,x)}{\partial y} = \Gamma^{-+}(y) - \Gamma^{--}(y)\, r(y,x) - r(y,x)\, \Gamma^{++}(y) + r(y,x)\, \Gamma^{+-}\, r(y,x). \tag{6a}$$

This is precisely the differential equation for the reflection operator obtained by Bellman, Kalaba, and Prestrud [7] using invariant imbedding methods. Proceeding in a similar fashion, we obtain the differential equations for the remaining operators; namely,

$$\frac{\partial r(x,y)}{\partial y} = t(y,x)\, \Gamma^{+-}(y)\, t(x,y) \tag{6b}$$

$$\frac{\partial t(y,x)}{\partial y} = t(y,x)\, (\Gamma^{+-}(y)\, r(y,x) - \Gamma^{++}(y)) \tag{7a}$$

$$\frac{\partial t(x,y)}{\partial y} = (-\Gamma^{++}(y) + r(y,x)\, \Gamma^{+-}(y))\, t(x,y) \tag{7b}$$

It should be noted at this point that eqn. (6b) couples r and t whereas eqn. (6a) only involves r. The differential equations involving rates of change with respect to the variable x can be obtained by adding a layer of thickness Δ to the x boundary, however, a simpler way is to interchange x and y. This is equivalent to inverting the layer.

Phase Function for a Spherical Polydispersion

To completely determine Γ^{++}, Γ^{--}, Γ^{+-}, and Γ^{-+} we need to consider the properties of a phase function which arises from a volume element containing either an assembly of identical particles that have a plane of symmetry and are randomly oriented, or a mixture of such assemblies. Since the phase function will only depend on the cosine of the angle between the directions (μ,ϕ) and (μ',ϕ') (see Fig. 2) then it will have the following symmetries:

$$P(\mu',\mu;\ \phi'-\phi) = P(\mu,\ \mu';\ \phi-\phi') \tag{8a}$$

$$P(-\mu',\ -\mu;\ \phi'-\phi) = P(\mu,\ \mu';\ \phi-\phi') \tag{8b}$$

$$P(-\mu,\ -\mu';\ \phi'-\phi) = P(\mu,\ \mu';\ \phi-\phi') \tag{8c}$$

$$P(\mu,\ \mu';\ \phi'-\phi) = P(\mu,\ \mu';\ \phi-\phi') \tag{8d}$$

$$P(\mu',\ \mu;\ \phi-\phi') = P(\mu,\ \mu';\phi-\phi') \tag{8e}$$

$$P(-\mu',-\mu;\ \phi-\phi') = P(\mu,\ \mu',\ \phi-\phi') \tag{8f}$$

$$P(-\mu,\ -\mu'\ ;\ \phi-\phi') = P(\mu,\ \mu',\ \phi-\phi') \tag{8g}$$

These are the phase function equivalents of the phase matrix symmetries presented by Hovenier.[6] Since eqn. (8d) implies that the phase function is an even function of the variable $\phi - \phi'$; we can therefore make the standard expansion into a Fourier series as follows:

$$P(\mu,\ \mu',\ \phi'-\phi) = \sum_{\ell=0}^{N} p_\ell(\mu,\ \mu')\ \cos\ \ell(\phi-\phi'). \tag{9}$$

Assuming the radiance has a similar expansion the equation of transfer can be decoupled into N + 1 independent equations and in our notation assumes the following form:

$$\left(\mu \frac{d}{d\tau} + 1\right) I_\ell^{+}(\tau;\ \mu,\mu_o) = \omega(\tau)\ \pi(1+\delta_{o\ell})\ \mathrm{x}$$
$$\left\{\int_o^1 p_\ell(\tau;\ \mu,\ \mu')\ I^{+}_\ell(\tau;\ \mu',\ \mu_o)\ d\mu' + \int_o^1 p_\ell(\tau;\ \mu,\ -\mu')\ I_\ell^{-}(\tau,\ \mu',\ \mu o)\ d\ \mu'\right\} \tag{10a}$$

$$\left(-\mu\frac{d}{d\tau} + 1\right) I_\ell^{-}(\tau;\mu,\mu_o) = \omega(\tau)\ \pi(1+\delta_{o\ell})\ \mathrm{x}$$
$$\left\{\int_o^1 p_\ell(\tau;-\mu,\mu')\ I_\ell^{+}(\tau,\mu',\mu_o)d\mu' + \int_o^1 p_\ell(\tau;-\mu,-\mu')\ I_\ell^{-}(\tau,\mu',\mu_o)d\mu'\right\} \tag{10b}$$

where $\delta_{o\ell}$ is the kronecker delta and

$$2\pi\int_{-1}^{1} p_o(\tau;\ \mu,\ \mu')d\mu' = 1 \tag{10c}$$

for all $0 < \mu \leq 1$. Discretizing the integrals by an appropriate quadrature with weights C_i and abscissas μ_i the r and t operators become matrices in this representation.

In particular

$$\underset{\sim}{\Gamma}_{\ell}^{++}(\tau) = \underset{\sim}{M}^{-1}\,[\underset{\sim}{E} - \omega(\tau)\,\pi\,(1+\delta_{o\ell})\,\underset{\sim}{p}_{\ell}^{++}(\tau)\,\underset{\sim}{C}] \tag{11a}$$

$$\underset{\sim}{\Gamma}_{\ell}^{+-}(\tau) = \underset{\sim}{M}^{-1}\,\omega(\tau)\pi(1+\delta_{o\ell})\,\underset{\sim}{p}_{\ell}^{+-}(\tau)\,\underset{\sim}{C} \tag{11b}$$

where

$$\underset{\sim}{p}_{\ell}^{++}(\tau) = \begin{bmatrix} p_{\ell}(\tau;\,\mu_1,\,\mu_1) & p_{\ell}(\tau;\,\mu_1,\,\mu_2)\ldots & p_{\ell}(\tau;\,\mu_1,\,\mu_m) \\ \vdots & & \vdots \\ p_{\ell}(\tau;\,\mu_m,\,\mu_1) & \ldots\ldots\ldots & p_{\ell}(\tau;\,\mu_m,\,\mu_m) \end{bmatrix}$$

$$\underset{\sim}{p}_{\ell}^{+-}(\tau) = \begin{bmatrix} p_{\ell}(\tau;\,\mu_1,\,-\mu_1)\ldots\ldots & p_{\ell}(\tau;\,\mu_1,\,-\mu_m) \\ \vdots & \vdots \\ p_{\ell}(\tau;\,\mu_m,\,-\mu_1)\ldots\ldots & p_{\ell}(\tau;\,\mu_m,\,-\mu_m) \end{bmatrix}$$

$$\underset{\sim}{C} = [C_j\delta_{jk}] \qquad\qquad \underset{\sim}{M} = [\mu_j\delta_{jk}] \tag{12}$$

Now m is the order of the quadrature used and by the symmetry relations in eqns. (8a) - (8g), $\underset{\sim}{p}_{\ell}^{+-}(\tau) = \underset{\sim}{p}_{\ell}^{-+}(\tau)$ and $\underset{\sim}{p}_{\ell}^{++}(\tau) = \underset{\sim}{p}_{\ell}^{--}(\tau)$. From this point on the subscript ℓ will be dropped since the equations are now decoupled and all symmetries will involve interchange of μ_i and μ_j which will be the same for every ℓ. It is also clear that the matrices $\underset{\sim}{p}_{\ell}^{++}(\tau)$ and $\underset{\sim}{p}_{\ell}^{+-}(\tau)$ are both symmetric.

Recognizing the fact that the Γ matrices are not symmetric we will now introduce the following new variables:

$$R(x,y) = M r(x,y) C^{-1} \tag{12a}$$

$$T(x,y) = M t(x,y) C^{-1} \tag{12b}$$

Using eqn. (6a) and pre and post multiplying it by M and C^{-1} respectively we get:

$$\begin{aligned}\frac{\partial R(x,y)}{\partial y} = {} & p^{+-}(y) - R(x,y)[M^{-1} - M^{-1}Cp^{++}(y)] \\ & - [M^{-1} - p^{++}(y)CM^{-1}]R(x,y) \\ & + R(x,y)CM^{-1}\ p^{+-}(y)CM^{-1}\ R(x,y)\end{aligned} \tag{13a}$$

The initial condition is that $R(x,x) = 0$. Next we will transpose eqn. (13a) and make use of the fact that $p^{+-}(y)$, $p^{++}(y)$, M^{-1}, and C are symmetric matrices. Doing this we get:

$$\begin{aligned}\frac{\partial \tilde{R}(x,y)}{\partial y} = {} & p^{+-}(y) - [M^{-1} - p^{++}(y)CM^{-1}]\tilde{R}(x,y) \\ & - \tilde{R}(x,y)[M^{-1} - M^{-1}C\ p^{++}(y)] \\ & + \tilde{R}(x,y)CM^{-1}\ p^{+-}(y)CM^{-1}\ \tilde{R}(x,y)\end{aligned} \tag{13b}$$

We thus see that $R(x,y)$ and $\tilde{R}(x,y)$ satisfy the same differential equation and have the same initial condition. Since the solution is unique we have

$$R(x,y) = \tilde{R}(x,y) \tag{14}$$

This is a true reciprocity relation since it involves interchange of source and receiver directions and holds for the general case of an inhomogeneous atmosphere.

Let us next perform the same transformation as above on eqns. (7a) and (7b). Doing this we obtain:

$$\frac{\partial \underset{\sim}{T}(y,x)}{\partial y} = \underset{\sim}{T}(y,x)\ \{\underset{\sim}{CM}^{-1}\ \underset{\sim}{p}^{+-}(y)\ \underset{\sim}{CM}^{-1}\ \underset{\sim}{R}(y,x)$$

$$- \underset{\sim}{R}(y,x)\underset{\sim}{M}^{-1} + \underset{\sim}{R}(y,x)\ \underset{\sim}{CM}^{-1}\ \underset{\sim}{p}^{++}(y)\} \qquad (15a)$$

$$\frac{\partial \underset{\sim}{T}(x,y)}{\partial y} = \{\underset{\sim}{R}(y,x)\ \underset{\sim}{CM}^{-1}\ \underset{\sim}{p}^{+-}(y)\ \underset{\sim}{CM}^{-1}$$

$$- \underset{\sim}{M}^{-1}\ \underset{\sim}{R}(y,x) + \underset{\sim}{p}^{++}(y)\ \underset{\sim}{CM}^{-1}\ \underset{\sim}{R}(y,x)\}\underset{\sim}{T}(y,x) \qquad (15b)$$

with initial condition $\underset{\sim}{T}(y,y) = \underset{\sim}{T}(x,x) = \underset{\sim}{E}$. Transposing eqn. (15b) we get:

$$\frac{\partial \tilde{\underset{\sim}{T}}(x,y)}{\partial y} = \tilde{\underset{\sim}{T}}(y,x)\ \{\underset{\sim}{CM}^{-1}\ \underset{\sim}{p}^{+-}(y)\ \underset{\sim}{CM}^{-1}\ \tilde{\underset{\sim}{R}}(y,x)$$

$$- \tilde{\underset{\sim}{R}}(y,x)\ \underset{\sim}{M}^{-1} + \tilde{\underset{\sim}{R}}(y,x)\ \underset{\sim}{CM}^{-1}\ \underset{\sim}{p}^{++}(y)\} \qquad (15c)$$

If we now compare eqn. (15c) and (15a) using eqn. (14), we find

$$\underset{\sim}{T}(y,x) = \tilde{\underset{\sim}{T}}(x,y) \qquad (16)$$

since they satisfy the same differential equation and have the same initial condition. This symmetry is a combination of time reversal and mirroring... the scattered beam by the meridian plane of incidence. It should be noted that no such relation as eqn. (16) exists for $\underset{\sim}{R}$ if the atmosphere is inhomogeneous. If the atmosphere is homogeneous then it is immediately clear that

$$\underset{\sim}{R}(x,y) = \underset{\sim}{R}(y,x) \ , \tag{17a}$$

$$\underset{\sim}{T}(x,y) = \underset{\sim}{T}(y,x) \ . \tag{17b}$$

It is only for this case that eqn. (16) implies exchange symmetry for T.

Symmetries in the Presence of a Reflecting Boundary

Assume that a reflecting layer exists on the bottom of an atmosphere. The reflection operator for the atmosphere and boundary will be denoted by $r_B(x,y)$ whereas the reflection operator for the boundary alone will be denoted by r_B. The equation for $r_B(x,y)$ is as follows (see PKC for details):

$$\underset{\sim}{r}_B(x,y) = \underset{\sim}{r}(x,y) + \underset{\sim}{t}(y,x)[\underset{\sim}{E} - \underset{\sim}{r}_B\, \underset{\sim}{r}(y,x)]^{-1} \underset{\sim}{r}_B \underset{\sim}{t}(x,y) \tag{18}$$

Employing the matrix representation and using the transformations in eqns. (12a) and (12b) we get:

$$\underset{\sim}{R}_B(x,y) = \underset{\sim}{R}(x,y) + \underset{\sim}{T}(y,x)\underset{\sim}{C}[\underset{\sim}{E} - \underset{\sim}{r}_B \underset{\sim}{M}^{-1} \underset{\sim}{R}(y,x)\underset{\sim}{C}]^{-1}$$

$$\times\ \underset{\sim}{r}_B \underset{\sim}{M}^{-1} \underset{\sim}{T}(x,y) \tag{19}$$

Transposing eqn. (19) we have:

$$\tilde{\underset{\sim}{R}}_B(x,y) = \tilde{\underset{\sim}{R}}(x,y) + \tilde{\underset{\sim}{T}}(x,y)\underset{\sim}{M}^{-1}\, \tilde{\underset{\sim}{r}}_B[\underset{\sim}{E} - \underset{\sim}{C}\, \tilde{\underset{\sim}{R}}(y,x)\underset{\sim}{M}^{-1} \tilde{\underset{\sim}{r}}_B]$$

$$\times\ \underset{\sim}{C}\, \tilde{\underset{\sim}{T}}(y,x) \tag{20}$$

Comparing eqns. (19) and (20) it can be seen that a necessary and sufficient condition for reciprocity to hold is that

$$\underset{\sim}{C}[\underset{\sim}{E} - \underset{\sim}{r}_B \underset{\sim}{M}^{-1} \underset{\sim}{R}(y,x)\underset{\sim}{C}]^{-1} \underset{\sim}{r}_B \underset{\sim}{M}^{-1}$$

$$= \underset{\sim}{M}^{-1} \tilde{\underset{\sim}{r}}_B[\underset{\sim}{E} - \underset{\sim}{C} \underset{\sim}{R}(y,x)\underset{\sim}{M}^{-1} \tilde{\underset{\sim}{r}}_B]^{-1} \underset{\sim}{C} \tag{21}$$

Performing a formal power series expansion on the inverse and equating the two sides of eqn. (21) term by term we must have

$$(\underset{\sim}{M} \underset{\sim}{C} \underset{\sim}{r}_B)^{tr} = \underset{\sim}{M} \underset{\sim}{C} \underset{\sim}{r}_B \tag{22}$$

where the superscript tr denotes matrix transpose.

There are two important surfaces which satisfy eqn. (22) namely, a Lambert surface and a specular surface. The reflection matrix for a Lambert surface is (see PKC)

$$\underset{\sim}{r}_B = k \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \end{bmatrix} \underset{\sim}{M}\underset{\sim}{C} \tag{23}$$

where k is a scalar constant depending on surface albedo. It is clear that $\underset{\sim}{r}_B$ satisfies eqn. (22) and hence reciprocity will hold. If the surface is a specular or a dielectric surface then $\underset{\sim}{r}_B$ will be a diagonal matrix and so will $\underset{\sim}{M}\underset{\sim}{C}\underset{\sim}{r}_B$ and eqn. (22) will again be satisfied.

Flux Conservation in a Conservative Scattering Inhomogeneous Atmosphere

In this section we will prove that flux is conserved in a conservative scattering inhomogeneous atmoshpere, i.e. $\omega_o(\tau) = 1.0$ for all depths, however, the phase function may vary with depth. We begin by defining the matrix Q as:

$$\underset{\sim}{Q} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{24}$$

This matrix has the effect of performing column sums on any matrix it premultiplies. The normalization of the phase function (see eqn. 10c) in the matrix representation reduces to

$$\underset{\sim}{Q}\underset{\sim}{C}(\underset{\sim}{p}^{++} + \underset{\sim}{p}^{+-}) = \underset{\sim}{Q} \tag{25}$$

where the factor of 2π has been omitted since it is irrelevant in this treatment. Rewriting this eqn. in terms of the $\underset{\sim}{\Gamma}$ matrices we get,

$$\underset{\sim}{Q}\underset{\sim}{C}(\underset{\sim}{E} - \underset{\sim}{M}\underset{\sim}{\Gamma}^{++} + \underset{\sim}{M}\underset{\sim}{\Gamma}^{+-})C^{-1} = \underset{\sim}{Q}$$

or (26)

$$\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}(\underset{\sim}{\Gamma}^{+-} - \underset{\sim}{\Gamma}^{++}) = \underset{\sim}{0}$$

Summing eqns. (6a) and (7a), premultiplying by QCM, and rearranging terms we get:

$$\begin{aligned}\frac{\partial}{\partial y}\{\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}[\underset{\sim}{r}(y,x) + \underset{\sim}{t}(y,x)]\} &= \underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}[\underset{\sim}{r}(y,x) + \underset{\sim}{t}(y,x)]\underset{\sim}{\Gamma}^{+-}(y)\underset{\sim}{r}(y,x) \\ &\quad -\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}[\underset{\sim}{r}(y,x) + \underset{\sim}{t}(y,x)]\underset{\sim}{\Gamma}^{++}(y) + \underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}[\underset{\sim}{\Gamma}^{+-}(y) - \underset{\sim}{\Gamma}^{++}(y)\underset{\sim}{r}(y,x)]\end{aligned} \tag{27}$$

Since the total energy escaping the system must equal the total energy entering the system we should expect that:

$$\underset{\sim}{Q}\underset{\sim}{C}\underset{\sim}{M}(\underset{\sim}{r}(y,x) + \underset{\sim}{t}(y,x) - \underset{\sim}{E}) = \underset{\sim}{0} \tag{28}$$

Using eqn. (26) and introducing the new variable $\underset{\sim}{Y}(y,x) = \underset{\sim}{QCM}[\underset{\sim}{r}(y,x) + \underset{\sim}{t}(y,x) - \underset{\sim}{E}]$, eqn. (27) can be written as:

$$\frac{\partial \underset{\sim}{Y}(y,x)}{\partial y} = \underset{\sim}{Y}(y,x)\underset{\sim}{\Gamma}^{+-}(y)\underset{\sim}{r}(y,x) - \underset{\sim}{Y}(y,x)\underset{\sim}{\Gamma}^{++}(y) \tag{29}$$

where $\underset{\sim}{Y}(y,y) = 0$.

Regarding $\underset{\sim}{r}(y,x)$ as a known function, the solution to eqn. (29) is:

$$\underset{\sim}{Y}(y,x) = 0. \tag{30}$$

Hence energy is conserved in a conservative scattering inhomogeneous atmosphere.

Conclusion

It has been demonstrated that the matrix operator theory affords a very powerful technique for analyzing symmetry relation under conditions of multiple scattering. The principles proven in this work will be invaluable in reducing computer time and storage for practical calculations. The general result obtained regarding various types of reflecting surfaces can be of great help in determining the type of surface seen from a remote sensing satellite viewing reciprocal points of a planet.

Acknowledgment

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C

References

1. G. N. Plass, G. W. Kattawar, and F. E. Catchings, Applied Optics (to appear).

2. S. Chandrasekhar, "Radiative Transfer," Dover Publ., New York (1960).

3. Z. Sekera, J. Opt. Soc. Amer., 56, 1732 (1966).

4. I. Busbridge, "The Mathematics of Radiative Transfer," Cambridge Univ. Press (1960).

5. M. F. A'Hearn, Ph.D. Thesis, University of Wisconsin, (1966).

6. J. W. Hovenier, J. Atmos. Sci., 26, 488 (1969).

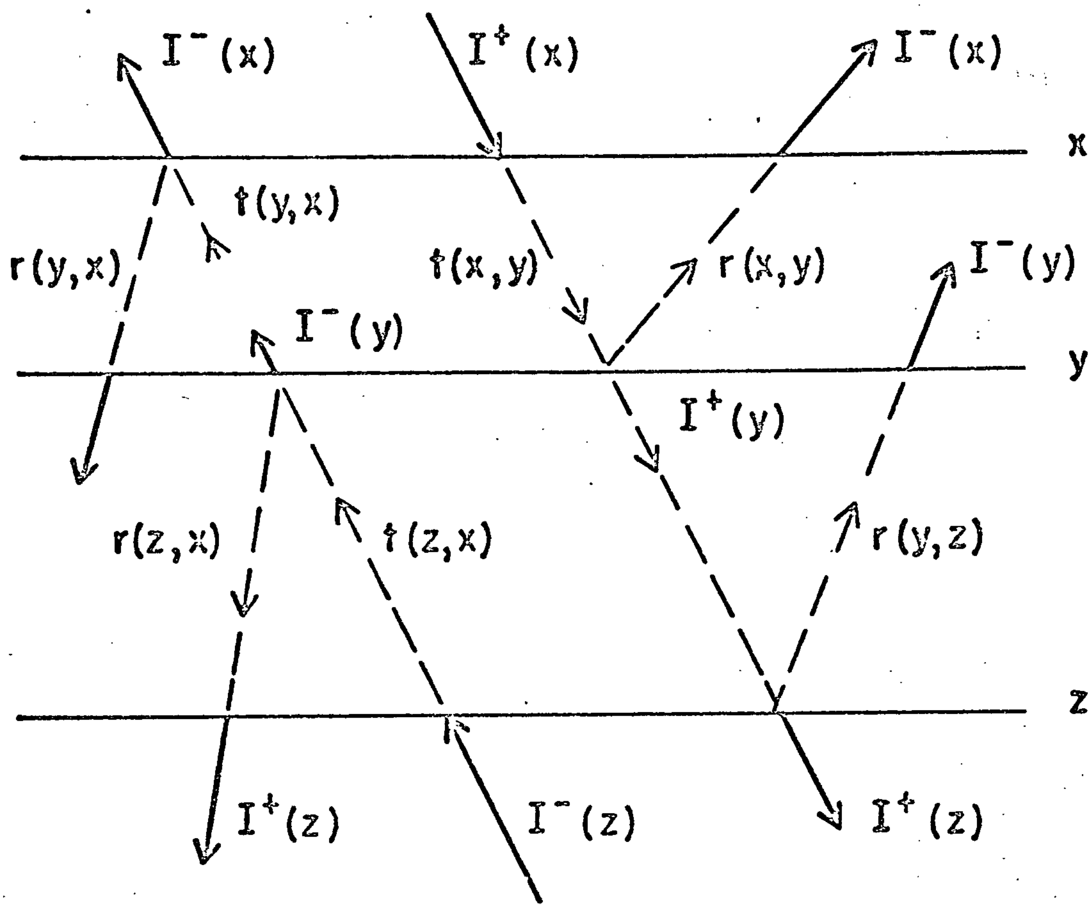
7. R. E. Bellman, R. E. Kalaba, and M. C. Prestrud, "Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness," American Elsevier Publishing Co., Inc., New York, (1963).

Figure Captions

Figure 1. A schematic representation of the reflection and transmission operators for a. addition of two inhomogeneous layers.
b. addition of an infinitesimal layer.

Figure 2. A schematic representation of a scattering process in order to visualize symmetry geometrically.

a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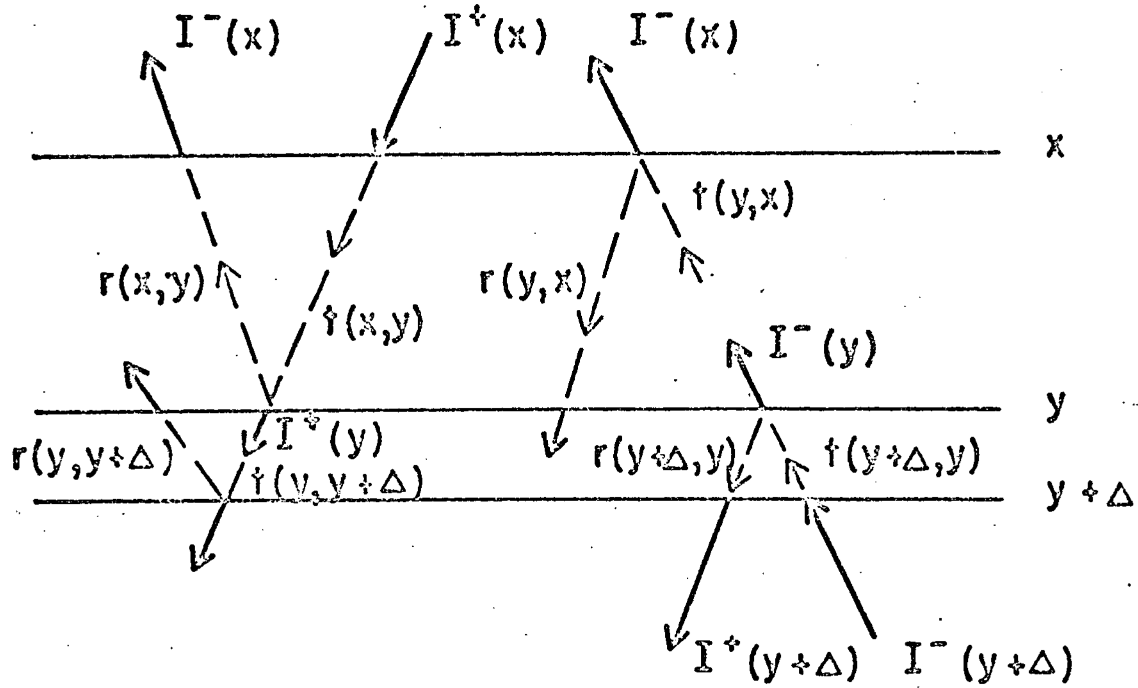
b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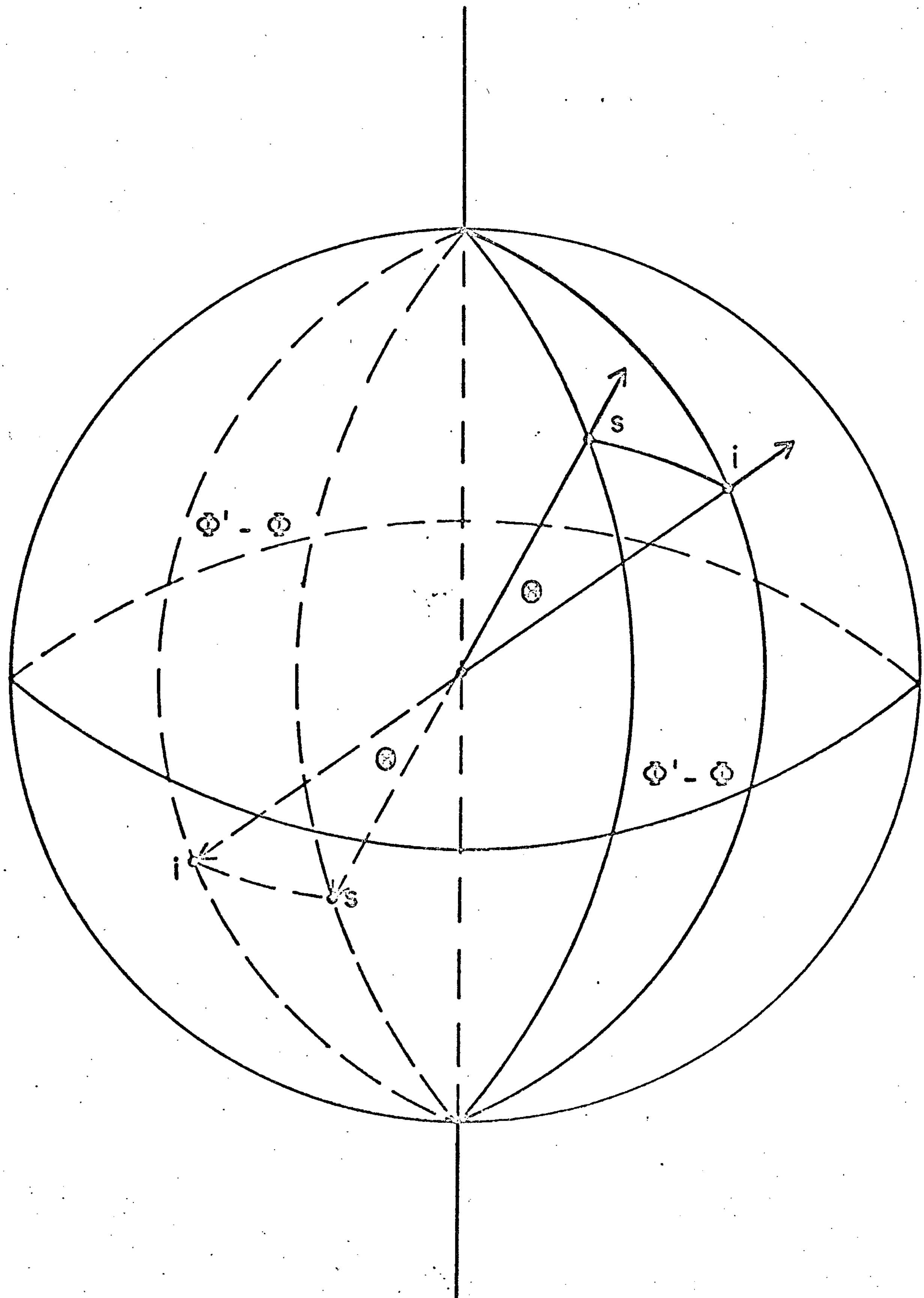
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